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Harmonic Hopf constructions and isoparametric gradient maps ☆

Tang Zizhou

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, PR China

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Abstract

This note provides a complete answer to a problem of Ding–Fan–Li on the homotopy classes of harmonic Hopf constructions. Moreover, it gives applications to isoparametric gradient maps.

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1. Introduction

Given a bi-eigenmap $\psi : S^p \times S^q \longrightarrow S^n$ with bi-eigenvalue (λ, μ), one defines a map $H_\alpha(\psi) : S^{p+q+1} \longrightarrow S^{n+1}$, called the α-Hopf construction on ψ, by

$$H_\alpha(\psi)(\sin\theta \cdot x,\ \cos\theta \cdot y) = \big(\sin\alpha(\theta) \cdot \psi(x, y),\ \cos\alpha(\theta)\big),$$

where one parametrizes the points of $S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R}^1$ by $(\sin\gamma \cdot z, \cos\gamma)$ with $z \in S^n$ and $0 \leqslant \gamma \leqslant \pi$; and $\alpha = \alpha(\theta) : [0, \pi/2] \longrightarrow [0, \pi]$ satisfies the boundary conditions

$$\alpha(0) = 0, \quad \alpha(\pi/2) = \pi. \tag{1.1}$$

It is clear that $H_\alpha(\psi)$ is homotopic to the original Hopf construction $H(\psi) = H_{\alpha_0}(\psi)$ in which $\alpha_0(\theta) = 2\theta$. It is well known [2] that $H_\alpha(\psi)$ is harmonic if and only if $\alpha = \alpha(\theta)$ satisfies the following pendulum type equation:

$$\ddot{\alpha} + (p \cdot \cot\theta - q \cdot \tan\theta)\dot{\alpha} - \left(\frac{\lambda}{\sin^2\theta} + \frac{\mu}{\cos^2\theta}\right)\sin\alpha \cdot \cos\alpha = 0 \tag{1.2}$$

with (1.1). Recently, Ding–Fan–Li [1] proved

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E-mail address: zztang@mx.cei.gov.cn.

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Theorem 1.1. *Let $p = 1$, $q > 1$, $\lambda \geqslant 1$ and $\mu > \lambda q$. Then Eqs. (1.1)–(1.2) have a solution $\alpha = \alpha(\theta)$ with $0 < \alpha(\theta) < \pi$ for $\theta \in (0, \pi/2)$.*

Remark. A. Gastel [3] proved independently the theorem by a different method.

The next task is to give various applications of Theorem 1.1. First recall that a bilinear map $\psi : \mathbb{R}^{t+1} \times \mathbb{R}^{r+1} \longrightarrow \mathbb{R}^{n+1}$ is called an orthogonal multiplication [2,5,6] if it is norm-preserving:

$$\left|\psi(x, y)\right| = |x| \cdot |y|$$

for all $x \in \mathbb{R}^{t+1}$ and $y \in \mathbb{R}^{r+1}$.

The following simple fact is observed by [2]:

Lemma 1.2. *Let $f : S^p \longrightarrow S^t$, $g : S^q \longrightarrow S^r$ be eigenmaps with eigenvalues λ, μ respectively. Let $\psi : \mathbb{R}^{t+1} \times \mathbb{R}^{r+1} \longrightarrow \mathbb{R}^{n+1}$ be an orthogonal multiplication. Then the composition $\psi(f, g) : S^p \times S^q \longrightarrow S^n$ is a bi-eigenmap with bi-eigenvalue (λ, μ).*

2. Application to Hopf fibrations

In Lemma 1.2, choose $f = \mathrm{id} : S^1 \longrightarrow S^1$; $g : S^{2m-1} \longrightarrow S^m$ the Hopf fibration ($m = 2, 4$ or 8), and let $\psi : \mathbb{R}^2 \times \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{m+2}$ be an orthogonal multiplication (the existence is trivial to see). Then, one gets a bi-eigenmap

$$\psi(f, g) : S^1 \times S^{2m-1} \longrightarrow S^{m+1}$$

with bi-eigenvalue $(1, 4m)$. By Theorem 1.1, its Hopf construction $H(\psi(f, g)) : S^{2m+1} \longrightarrow S^{m+2}$ is homotopic to a harmonic map. Ding–Fan–Li [1] posed naturally the following

Problem. Determine its homotopy class in the corresponding homotopy group.

One of the main purposes of this note is to provide a complete answer to this problem. For it, we first need some preliminaries. Given two maps of spheres $f : S^p \longrightarrow S^t$ and $g : S^q \longrightarrow S^r$, recall that the join $f * g$ of f and g [9] is a map $f * g : S^{p+q+1} \longrightarrow S^{t+r+1}$ defined by

$$(f * g)(\sin\theta \cdot x, \cos\theta \cdot y) = \bigl(\sin\theta \cdot f(x), \cos\theta \cdot g(y)\bigr)$$

where $0 \leqslant \theta \leqslant \pi/2$, $x \in S^p$, $y \in S^q$. Similarly one has the definition of α-join for a given function $\alpha = \alpha(s) : [0, \pi/2] \longrightarrow [0, \pi/2]$ with $\alpha(0) = 0$, $\alpha(\pi/2) = \pi/2$. We now state

Lemma 2.1. *There is a homotopy between $H(\psi(f, g))$ and $H(\psi) \circ (f * g)$ for two maps $f : S^p \longrightarrow S^t$, $g : S^q \longrightarrow S^r$ and an orthogonal multiplication $\psi : \mathbb{R}^{t+1} \times \mathbb{R}^{r+1} \longrightarrow \mathbb{R}^{n+1}$.*

Proof. By a direct verification [9]. □

The Ding–Fan–Li problem is just to determine the homotopy class $H(\psi(f, g))$ in the homotopy group $\pi_{2m+1} S^{m+2}$ for $m = 2, 4$ or 8.

According to Lemma 2.1, we need only to investigate the Hopf construction $H(\psi) : S^{m+2} \longrightarrow S^{m+2}$ for an orthogonal multiplication $\psi : \mathbb{R}^2 \times \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{m+2}$.

It is well known (Hopf theorem) that the Brouwer degree of a map $\Phi : S^{m+2} \longrightarrow S^{m+2}$ characterizes its homotopy class $[\Phi]$ in the homotopy group $\pi_{m+2} S^{m+2} \cong \mathbb{Z}$.

Lemma 2.2. $\deg H(\psi) = 0$.

Proof. $H(\psi)$ is the original Hopf construction, hence an even map, i.e.

$$H(\psi) = H(\psi) \circ A$$

where $A : S^{m+2} \longrightarrow S^{m+2}$ is the antipodal map. It follows immediately that

$$\deg H(\psi) = \deg H(\psi) \cdot \deg A = -\deg H(\psi),$$

since S^{m+2} has even dimension. Therefore we have $\deg H(\psi) = 0$, which completes the proof. □

Summarizing arguments above, we get at last the answer to Ding–Fan–Li problem.

Theorem 2.3. *The Hopf construction $H(\psi(f, g))$ is homotopic to a constant map, where $f = \mathrm{id} : S^1 \longrightarrow S^1$, $g : S^{2m-1} \longrightarrow S^m$ is the Hopf fibration ($m = 2, 4$ or 8), and $\psi : \mathbb{R}^2 \times \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{m+2}$ is an orthogonal multiplication.*

Remark. The Hopf construction $H(\psi(f, g)) : S^{2m+1} \longrightarrow S^{m+2}$ gives rise to a map from the real projective space $\mathbb{R}P^{2m+1} = S^{2m+1}/\mathbb{Z}_2$, which is still homotopic to a harmonic map. It will be interesting [4] to investigate its homotopy class in the corresponding homotopy set $[\mathbb{R}P^{2m+1}, S^{m+2}]$. Since $2m + 1 \leqslant 2(m + 2) - 1$, the homotopy set $[\mathbb{R}P^{2m+1}, S^{m+2}]$ admits an Abelian group structure.

3. Application to isoparametric gradient maps

Another interesting occurrence is to apply Theorem 1.1 to isoparametric gradient maps. Fortunately we can get several harmonic maps between spheres which are homotopically non-trivial in this case.

Let f be a homogeneous polynomial of degree k on the Euclidean space $\mathbb{R}^{n+2}$. f is called an isoparametric polynomial if it satisfies the following Cartan–Münzner's differential equations:

$$\begin{cases} |\nabla f|^2 = k^2 |x|^{2k-2}, \\ \Delta f = \frac{1}{2} k^2 (m_2 - m_1) |x|^{k-2} \end{cases} \tag{3.1}$$

where ∇f and Δf denote the gradient and the Laplacian of f respectively, and m_1, m_2 are two (possibly equal) natural numbers with $n = \frac{1}{2}(m_1 + m_2)k$.

In order to construct harmonic maps between spheres, from now on, let $m_1 = m_2 = m$, $k \geqslant 3$, and let

$$\Phi_k(m) = \frac{1}{k} \nabla f : S^{n+1} \longrightarrow S^{n+1}$$

be the gradient map. The Brouwer degree of $\Phi_k(m)$ was calculated in [7] (in fact, the Brouwer degree of $\frac{1}{k}\nabla f$ was calculated in [7] even if $m_1 \neq m_2$).

Theorem 3.1.

$$(1)\quad \deg \Phi_3(m) = \begin{cases} 0 & \textit{for } m = 1; \\ -2 & \textit{for } m = 2, 4 \textit{ or } 8. \end{cases}$$

$$(2)\quad \deg \Phi_4(m) = \begin{cases} 1 & \textit{for } m = 1; \\ -3 & \textit{for } m = 2. \end{cases}$$

$$(3)\quad \deg \Phi_6(m) = \begin{cases} 1 & \textit{for } m = 1; \\ -5 & \textit{for } m = 2. \end{cases}$$

We now choose $f = \mathrm{id} : S^1 \longrightarrow S^1$ and $g = \Phi_k(m) : S^{km+1} \longrightarrow S^{km+1}$ with even km. Let $\psi : \mathbb{R}^2 \times \mathbb{R}^{km+2} \longrightarrow \mathbb{R}^{km+2}$ be an orthogonal multiplication via the multiplication of complex numbers. It follows from Lemma 1.2 that

$$\psi(f, g) : S^1 \times S^{km+1} \longrightarrow S^{km+1}$$

is a bi-eigenmap with bi-eigenvalue $(\lambda, \mu) = (1, (k - 1)(k - 1 + km))$. Thus, by Theorem 1.1, its Hopf construction

$$H\big(\psi(f, g)\big) : S^{km+3} \longrightarrow S^{km+2}$$

is homotopic to a harmonic map, since $(k-1)(k-1+km) > km+1$. The natural task is to determine the homotopy class in the homotopy group $\pi_{km+3}S^{km+2}$, which is isomorphic to $\mathbb{Z}_2$ for $km > 0$.

By virtue of Lemma 2.1, $H(\psi(f,g))$ is homotopic to $H(\psi) \circ (f * g)$. Moreover, $f * g$ is homotopic to $\Sigma^2\Phi_k(m): S^{km+3} \longrightarrow S^{km+3}$, where Σ^2 denote the 2-times iterated suspension. Clearly

$$\deg\big(\Sigma^2\Phi_k(m)\big) = \deg \Phi_k(m).$$

Thus, we are left to determine the homotopy class

$$H(\psi) \in \pi_{km+3}S^{km+2} \cong \mathbb{Z}_2.$$

To do it, we have more general assertions.

Lemma 3.2. *Let $\psi: \mathbb{R}^2 \times \mathbb{R}^{2p} \longrightarrow \mathbb{R}^{2p}$ be an orthogonal multiplication. Then the Hopf construction $H(\psi): S^{2p+1} \longrightarrow S^{2p}$ is surjective.*

Proof. Recall first that $H(\psi): S^{2p+1} \longrightarrow S^{2p}$ is given by

$$(x,y) \longmapsto \big(2\psi(x,y), |x|^2 - |y|^2\big)$$

for $(x,y) \in S^{2p+1} \subset \mathbb{R}^2 \times \mathbb{R}^{2p}$. Given any point (z,t) in $S^{2p} \subset \mathbb{R}^{2p} \times \mathbb{R}^1$, we aim to find a point (x,y) in S^{2p+1} with $H(\psi)(x,y) = (z,t)$. If $|z| = \sqrt{1-t^2} = 0$, it is trivial. We now assume that $|z| > 0$. Choose $x = (x_1, 0)$ with $x_1 = \sqrt{\frac{1+t}{2}} > 0$. Clearly $|x|^2 - |y|^2 = 2|x|^2 - 1 = t$. Thus it suffices to find $y \in \mathbb{R}^{2p}$ with $|y| = \sqrt{1-|x|^2}$ and $2\psi(x,y) = z$.

Since

$$\left|\frac{z}{2|x|}\right| = \sqrt{1-|x|^2},$$

we need only to find $y \in \mathbb{R}^{2p}$ such that it satisfies $\psi(\frac{x}{|x|}, y) = \frac{z}{2|x|}$. However, ψ is an orthogonal multiplication, for $\frac{x}{|x|} \in S^1 \subset \mathbb{R}^2$, $\psi(\frac{x}{|x|}, \cdot): \mathbb{R}^{2p} \longrightarrow \mathbb{R}^{2p}$ is an orthogonal transformation. The existence of such y follows immediately. $\square$

Lemma 3.3. *Let $\psi: \mathbb{R}^2 \times \mathbb{R}^{2p} \longrightarrow \mathbb{R}^{2p}$ $(p \geqslant 2)$ be an orthogonal multiplication. Then $[H(\psi)] \in \pi_{2p+1}S^{2p} \cong \mathbb{Z}_2$ is given by*

$$\big[H(\psi)\big] = \begin{cases} 1 & \text{for } p \text{ odd;} \\ 0 & \text{for } p \text{ even.} \end{cases}$$

Proof. Put Lemma 3.2 and Theorem 1.2 in [5] together. $\square$

At last, we arrive at

Corollary 3.4. *Each of the homotopy group $\pi_7S^6 \cong \mathbb{Z}_2$, $\pi_{11}S^{10} \cong \mathbb{Z}_2$ and $\pi_{15}S^{14} \cong \mathbb{Z}_2$ admits a harmonic generator via the isoparametric gradient maps.*

Proof. Consider the gradient map of the isoparametric polynomial with $(k,m) = (4,1), (4,2)$ and $(6,2)$ respectively. $\square$

Remark. Compare the results with [5–8] where quite different methods were used.

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